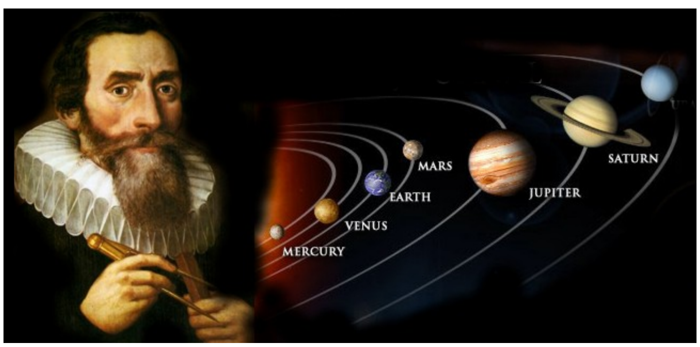
2.2: The Elliptical Universe

Elliptical Equations and Graphs

Ellipticals are the most numerous type of galaxy. Elliptical galaxies - named because of their shapes - range from circular (remember, a circle is an ellipse!) to long, narrow, and cigar-shaped. Elliptical galaxies are denoted by the letter E. They are also given a number from 0 to 7. An E0 galaxy looks like a circle. An E7 galaxy is very long and thin.

Photo Credit: [http://www.ultimateuniverse.net/galaxies.html]

Driving Question: How can we write an equation to define the elliptical motion of celestial bodies?



Part 1: Elliptical Orbits

In 1571, German mathematician Johannes Kepler discovered that the planets revolved around the sun in an elliptical orbit, not a circular one. Learn more about Kepler's findings on The Science Channel's "100 Greatest Discoveries." Watch the video at http://science.howstuffworks.com/29276-100-greatest-discoveries-elliptical-orbits-video.htm.

1. Why might people have believed in circular orbits?

2. Animated Solar System: Follow the step-by-step instructions below to create an animated model of the solar system using GeoGebra
   a. Open New GeoGebra file in Graphics view with Axes on
   b. Set window as follows [Edit > Object Properties > Preferences - Graphics]:
      i. x-min: -16
      ii. x-max: 9
      iii. y-min: -8
      iv. y-max: 8

3. Construct the Sun at the origin
   a. Enter: (0,0)
   b. Enter: $x^2 + y^2 = 1$
   c. Color the sun yellow: Select circle [Edit > Object Properties > Color > Select color > move Opacity Slider to 100%]
   d. Change the name of this point to *O* (the letter O for Origin) and the Caption to Sun: Right-click on point > Object Properties. Also check "Show Label" but change drop-down to "Caption."

4. Construct animation slider: Select **Slider Tool** and click in the upper left corner of the screen. In the pop-up window set properties: *Name*: n. *Interval*: (0,3600), *Increment* 0.1; *Animation*: speed 0.01 to get slow movement of the planets, *Repeat*: increasing.

5. Construct Mercury's elliptical Orbit
   a. Enter: $\frac{(x+0.5)^2}{5} + \frac{y^2}{1.5} = 1$
   b. Object Properties > Caption: *Mercury Orbit*
      i. Do not display the caption; this is just to keep organized.

6. Construct planet Mercury
   a. Construct the 2 points where Mercury: change Point button to **Intersect** (click on drop down arrow on the bottom right of the tool button)
      i. Select *Mercury Orbit* then Select x-axis.
      ii. Two new points are constructed *A* and B
   b. Select the tool **Angle with Given Size** and use it by choosing the points *A* and then *O* as the first side of the angle.
      i. Replace the angular measure of 45 degrees by the variable *n* (remember you named the slider n.) A new point is created, *A*' and its position is now linked to the slider value.
         1. Move the slider to adjust the position of the new point to an open area
         2. Hide the angle α: Select the angle (or measurement) > Edit > Object Properties > Deselect "Show Object"

c. Define the center of the planet: Construct ray OA' using the tool **Ray through Two Points**. (Make sure the ray attaches to *A'*)
    i. Construct the intersection point of the segment with the ellipse.
    ii. Rename the point P1 and add a caption: *Mercury*.
d. Construct the planet represented by a circle. Select the tool **Circle with Centre and Radius**, click on point *P'* (Mercury), enter 0.2 for the radius and click OK. The circle will be displayed. Use Object Properties to select the desired color and transparency.
    i. Hide ray OA'
    ii. Hide the points *A*, *B* and *A'*
e. Test the movement of the planet by selecting in the slider the option Animation On.

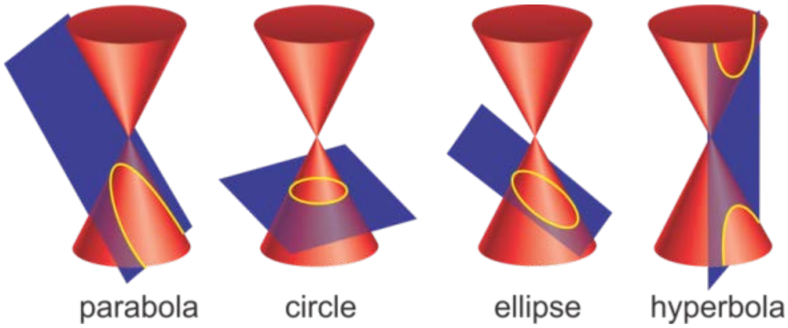
7. Create Venus and its elliptical orbit:
    a. Orbit: $\frac{(x+0.5)^2}{7.5} + \frac{y^2}{2.25} = 1$
    b. Planet: Follow step 5 above, except enter 0.9n for the angle measurement (Venus is slower than Mercury)
        i. Angular measure\
        ii. Radius: 0.25

8. Continue in the same way to construct the other planets adding measurements to *n* insures that the planets are not in a straight line and they orbit.
    a. Earth
        i. Orbit: $\frac{(x+0.5)^2}{11.25} + \frac{y^2}{3.375} = 1$
        ii. Angular measure: 0.8 n°+45°
        iii. Radius: .25
    b. Mars
        i. Orbit: $\frac{(x+0.5)^2}{16.25} + \frac{y^2}{5} = 1$
        ii. Angular measure: 0.7 n°
        iii. Radius: .2
    c. Jupiter
        i. Orbit: $\frac{(x+0.5)^2}{32.5} + \frac{y^2}{10} = 1$
        ii. Angular measure: 0.5 n° +35°
        iii. Radius: .65
    d. Saturn
        i. Orbit: $\frac{(x+0.5)^2}{65} + \frac{y^2}{20} = 1$
        ii. Angular measure: 0.3n°+70°
        iii. Radius: .5

e. Uranus
   i. Orbit: $\frac{(x+0.5)^2}{150} + \frac{y^2}{40} = 1$
   ii. Angular measurement: 0.2 n°+20°
   iii. Radius: .4
f. Neptune
   i. Orbit: $\frac{(x+0.5)^2}{300} + \frac{y^2}{70} = 1$
   ii. Angular measurement: 0.1 n°
   iii. Radius: .4
g. Comet (Extra Credit)
   i. Orbit: $\frac{(x-16)^2}{400} + \frac{y^2}{36} = 1$
   ii. Angular measurement: 0.4 n°
   iii. Radius: hmmm… don't they change size as they approach the sun

Customize your design and share with the class!

Part 2a: Investigating Ellipses



Remember that an ellipse is the curves we get when we make a straight cut in a cone, as shown in the figure above.

Open the interactive conic section located at http://ggbtu.be/mqZ8aGDzR and move the sliders to create an ellipse.

1. Describe, in detail, what conditions result in an Ellipse

Open the interactive ellipse located at http://ggbtu.be/mzciTMVPq and move the sliders to create differently shaped ellipses.

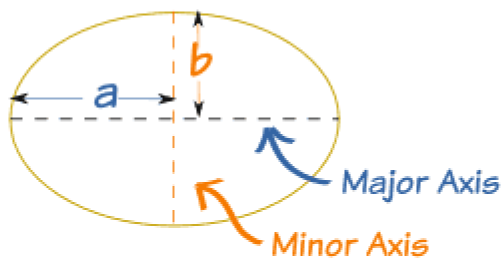
2. In your own words, what is the focus of an ellipse?

3. What do you notice about the measurements of the two lines connecting to the pencil?

4. Define an ellipse in terms of its foci (plural of focus): "An ellipse is the set of points ____."

Part 2b: The Equation of an Ellipse

Two important measurements in an ellipse are the two axes: The **Major Axis** is the longest diameter. It goes from one side of the ellipse, through the center, to the other side, at the widest part of the ellipse. And the **Minor Axis** is the shortest diameter (at the narrowest part of the ellipse).

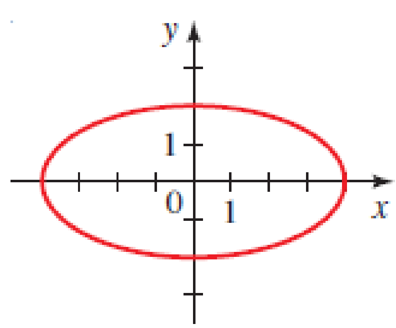


The **Semi-major Axis** is half of the Major Axis, and the **Semi-minor Axis** is half of the Minor Axis – these are also referred to as the radii of an ellipse.

By placing an ellipse on a coordinate plane (with its major axis on the x-axis and minor axis on the y-axis), the equation of the curve is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

5. What is the equation of the following ellipse?



Open the interactive app located at http://ggbtu.be/mY3bA0QXd and move the sliders to match the shape of the stadium.

6. What is the equation that matches the Olympic Stadium photo?

7. What conditions result in an ellipse that stretches vertically instead of horizontally?
8. Predict the standard form equation of an ellipse that is centered at (h, k).

2c. Finding the Foci… and more!

Example: Find the foci of the ellipse $16x^2 + 9y^2 = 144$

Solution: First we put the equation in standard form. Dividing by 144, we get

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

To find a and b, take the square root of each denominator: $a = 4$ and $b = 3$. (We always assign a to the larger radius.)

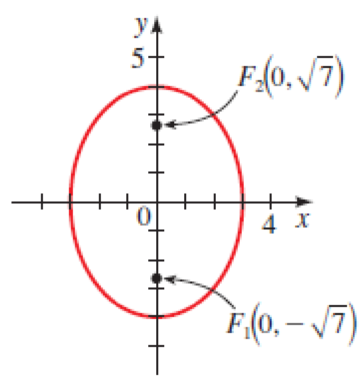
In this case the major axis is vertical because the y-radius (4), is larger than the x-radius (3). In order to find the locations of the two foci, we will need to find the focal radius represented as c using the following relationship: $a^2 - b^2 = c^2$.

$$4^2 - 3^2 = c^2$$

$$16 - 9 = c^2$$

$$7 = c^2$$

$$\sqrt{7} = c$$



Thus the Foci (foh-cee) are $\sqrt{7}$ units above and below the center, which, in this case, is at (0,0). So, we write the foci as $F(0, \pm\sqrt{7})$.

The other common features to locate are the **vertices**, which in this case are also x- and y-intercepts. We just list the as coordinate pairs as follows: (0,±4) and (±3,0).

Your Turn:

9. Find the foci of the ellipse $4x^2 + 25y^2 = 100$

2d. Finding the Equation of an Ellipse

Example: The vertices of an ellipse are (±4,0) , and the foci are (±2,0). Find its equation, and sketch the graph.

Solution: Since the vertices are (±4,0), we have $a = 4$ and the major axis is horizontal.

The foci are (±2,0), so c=2. To write the equation, we need to find b. Since we have

$$2^2 = 4^2 - b^2$$

$$b^2 = 16 - 4 = 12$$

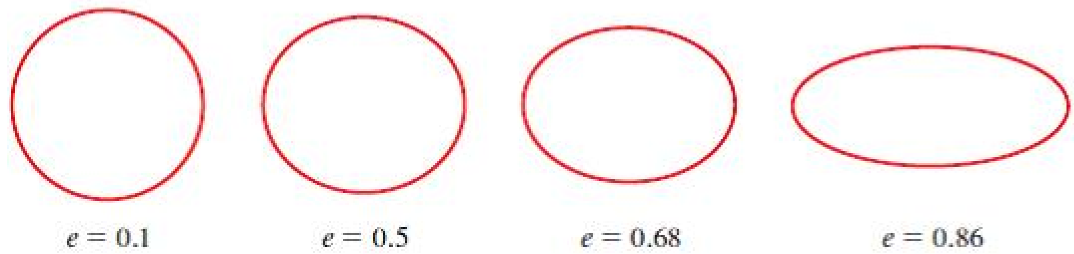
Therefore, the equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

Your Turn:

10. The vertices of an ellipse are (0,±5) , and the foci are (0,±3). Find its equation, and sketch the graph.

2e. Eccentricity

We measure the deviation of an ellipse from being circular by the ratio of a and c.



Thus if e is close to 1, then c is almost equal to a, and the ellipse is elongated in shape, but if e is close to 0, then the ellipse is close to a circle in shape. The eccentricity is a measure of how "stretched" the ellipse is.

Example: Find the equation of the ellipse with foci $(0,\pm 8)$ and eccentricity $e = \frac{4}{5}$, and sketch its graph.

Solution: We are given and $c = 8$. Thus

$$\frac{4}{5} = 8a$$

$$\frac{4}{5} = \frac{8}{a}$$

$$4a = 40$$

$$a = 10$$

To find b, we use the fact that $a^2 - b^2 = c^2$

$$10^2 - b^2 = 8^2$$

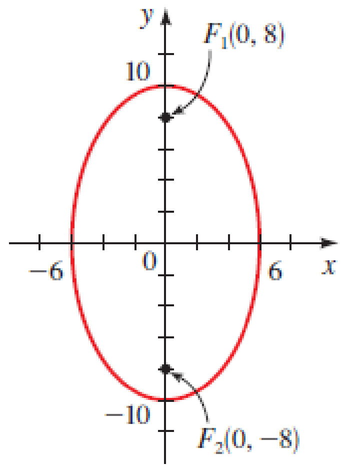
$$100 - 64 = b^2$$

$$36 = b^2$$

$$6 = b$$

Thus the equation of the ellipse is $\frac{x^2}{36} + \frac{y^2}{100} = 1$

Because the foci are on the y-axis, the ellipse is oriented vertically. To sketch the ellipse, we find the intercepts: The x-intercepts are ± 6, and the y-intercepts are ± 10. The graph is sketched below.



<u>Your Turn</u>:

11. Find the equation of the ellipse with foci $(0,\pm 2)$ and eccentricity $e = \frac{1}{9}$, and sketch its graph.

Eccentricities of the Orbits of the Planets

The orbits of the planets are ellipses with the sun at one focus. For most planets these ellipses have very small eccentricity, so they are nearly circular. However, Mercury and Pluto, the inner-most and outermost known planets, have visibly elliptical orbits.

Planet	Eccentricity
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.056
Uranus	0.046
Neptune	0.010
Pluto	0.248